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Approximation Based on Nonscalar Observations

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INTRODUCTION

There are situations in which one wants to approximate a mathematical object x in terms of mathematical observations of x. For example, x may be a function on $[0, 1]$ to $\mathbb{R}$, and the observations may be a table of values of x at $s_1, s_2, \ldots, s_m \in [0, 1]$. Here the observations are the m-tuple $x(s_1), \ldots, x(s_m)$. As another example, x may be a function on the square $[0, 1] \times [0, 1]$ to $\mathbb{R}$, and the observations may be specified cross-sections of x, that is, the functions $x(s_1, \cdot), \ldots, x(s_m, \cdot)$, and $x(\cdot, t_1), \ldots, x(\cdot, t_{m'})$ on $[0, 1]$ to $\mathbb{R}$, where $s_i, t_j \in [0, 1]$. In this instance the observations are functions and, therefore, contain more information than a finite number of scalars.

Whether the observations of x are scalars or not, it is advantageous to combine them into one object which we denote Fx and which we call the observation of x. In the second example above, Fx is the $(m + m')$-tuple of functions $x(s_1, \cdot), \ldots, x(s_m, \cdot); x(\cdot, t_1), \ldots, x(\cdot, t_{m'})$.

Besides approximating x, we may wish to approximate Gx, where G is a given operator, in terms of the observation Fx. Let us write

$$Ax = EFx,$$

where E is the operator which describes our operation on the observation and A is the operator which describes the process of approximation. The error in the approximation of Gx by Ax is

$$Rx = Gx - Ax = Gx - EFx.$$

This paper is concerned with the choice of E and the study of R.

Suppose that the operators G, F, and E have been specified. Whereas, Ax is determined by Fx alone, the error Rx need not be so determined. We will posit the existence of a coobservation operator U such that Ux and Fx

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together determine x and, therefore, Gx and Rx. Then partial information about the coobservation Ux, for example a bound on the norm of Ux, will permit us to appraise Rx.

The optimal approximation, in senses to be described, of x is called the spline approximation of x, relative to F and U. And the optimal approximation of Gx is simply $G\xi$, where ξ is the spline approximation of x.

In particular cases Fx and Ux will not merely determine x but will contain more information than is necessary. Such excess information permits us to draw conclusions about Rx which are not otherwise available. For example, in our first illustration above, if $Ux = x_n =$ the nth derivative of x on $[0, 1]$, where $n \leqslant m$, then the smaller n is, the more valuable is knowledge about x_n. If $n < m$, the observation $x(s_1), \ldots, x(s_m)$ and the coobservation x_n overdetermine x.

The present paper continues the work of my earlier [11], but treats the entire problem anew. Hypotheses are reduced. No conditions of completeness are required. Proofs are improved, and two errors are corrected.[1]

Splines, as defined in this paper, include all minimizing splines of other authors. Our hypotheses of linearity and inner products are justified, I think, by their power and by the fact that many preproblems can be put into our mold as easily as into another. A vague space which is part of a preproblem can often be exemplified by an inner product space [9, Chapter 9; 10].

Applications of the present theory are in Sections 8 and 9 below. In Section 8 the splines are harmonic functions and the observations are the boundary values of the unknown functions. Our theorems then pertain to the Dirichlet problem.

1. Hypotheses

Let X be a linear space, Y, Z be inner product spaces, and W be a normed linear space, all over the real or the complex numbers. Let

$$F: X \to Y, \qquad U: X \to Z, \qquad G: X \to W$$

be linear operators. For any $x \in X$, we call Fx the *observation* of x and Ux the *coobservation*. We wish to approximate Gx by Ax, where Ax depends only on Fx. In particular, W may be X, and G the identity.

Assume that

$$Fx = 0, \qquad Ux = 0, \qquad x \in X$$

imply that $x = 0$. Thus, Fx and Ux together determine $x \in X$.

[1] On page 228 of [11], the hypothesis that F^iX is closed does not imply that $V\mathscr{X}$ is closed. Lemma 4, which is true, is misapplied to a space which need not be a Hilbert space.

2. The Hilbert Space $\mathscr{X}$ and the Splines

Following Golomb–Weinberger [5], who, however, consider only the scalar case (Y finite dimensional), we now construct a Hilbert space $\mathscr{X}$. For $x, y \in X$, put

$$(x, y) = (Fx, Fy) + (Ux, Uy),$$

where the terms on the right are inner products on Y and Z, respectively. Then $(x, x) = 0$ iff $x = 0$. Therefore, (x, y) is a valid inner product on X. Let $\mathscr{X}$ denote the space X with this inner product. Then the operators F and U are bounded on $\mathscr{X}$ and, hence, continuous on $\mathscr{X}$.

Take completions $\bar{\mathscr{X}}$, $\bar{Y}$, $\bar{Z}$ of the spaces $\mathscr{X}$, Y, Z, and completions $\bar{F}$, $\bar{U}$ of the operators F, U, respectively [9, p. 302]. Then $\bar{\mathscr{X}}$, $\bar{Y}$, $\bar{Z}$ are Hilbert spaces and $\bar{F}$, $\bar{U}$ are linear and continuous. We continue to call $\bar{F}$ the observation and $\bar{U}$ the coobservation operator. Note that

$$\bar{F}x = 0, \qquad \bar{U}x = 0, \qquad x \in \bar{\mathscr{X}}$$

imply that $x = 0$, and that

$$(x, y) = (\bar{F}x, \bar{F}y) + (\bar{U}x, \bar{U}y), \qquad x, y \in \bar{\mathscr{X}}.$$

Put

$$N = \text{kernel}\,\bar{F} = \{x \in \bar{\mathscr{X}} : \bar{F}x = 0\}$$

and

$$M = N^{\perp} = \{x \in \bar{\mathscr{X}} : (x, \zeta) = 0 \text{ whenever } \zeta \in N\}.$$

Now if $\zeta \in N$, then $\bar{F}\zeta = 0$ and $(x, \zeta) = (\bar{U}x, \bar{U}\zeta)$. Hence,

$$M = \{x \in \bar{\mathscr{X}} : (\bar{U}x, \bar{U}\zeta) = 0 \text{ whenever } \zeta \in N\}$$

and

$$M \supset \text{kernel}\,\bar{U}. \tag{1}$$

We call M the space of *splines*, relative to F, U.

Assume that neither M nor N is the entire space $\bar{\mathscr{X}}$; that is, assume that for some $x^0 \in X$, $Fx^0 \neq 0$; and for some $y^0 \in X$, both $Fy^0 = 0$ and $y^0 \neq 0$.

Put

$$\prod = \text{Proj}_M = \text{orthogonal projection of } \bar{\mathscr{X}} \text{ onto } M.$$

For $x \in \bar{\mathscr{X}}$, we call $\prod x$ the *spline approximation* of x. The error in the approximation of x by $\prod x$ is

$$x - \prod x = \text{Proj}_N x.$$

Note that the splines and other constructions of this section are independent of W and G.

3. The Quotient Theorem

The quotient theorem is fundamental in linear approximation and is as follows [7; 9, p. 310].

Let $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ be normed, linear spaces, $\mathscr{A}$ and $\mathscr{C}$ being complete. Let P be a surjective linear continuous operator on $\mathscr{A}$ onto $\mathscr{B}$, and R be a linear continuous operator on $\mathscr{A}$ into $\mathscr{C}$. If $Rx = 0$ whenever $Px = 0$, $x \in \mathscr{A}$, then there is a unique operator Q on $\mathscr{B}$ to $\mathscr{C}$ such that $R = QP$. And Q (the quotient of R by P) is linear with closed graph. Furthermore, Q is continuous if $\mathscr{B}$ is complete.

Completeness of $\mathscr{B}$ is sufficient but not necessary for the continuity of Q. Of course, $\mathscr{B}$ is surely complete if $\mathscr{B}$ is finite dimensional.

Lemma 1 (dependence of error on coobservation). *There is a unique map $q: \bar{U}\mathscr{X} \to \mathscr{X}$ such that*

$$\mathrm{Proj}_N = q\bar{U};$$

q is linear and continuous, with Banach norm unity.

Proof. Apply the quotient theorem with $\mathscr{A} = \mathscr{X} = \mathscr{C}$ and $\mathscr{B} = \bar{U}\mathscr{X}$. If $\bar{U}x = 0$, $x \in \mathscr{X}$, then $x \in M$ by (1) and $\mathrm{Proj}_N x = 0$. Thus, q exists and is unique and linear.

It remains to show that the Banach norm $\|q\|$ of q is unity, as this will imply that q is continuous. Now

$$\|q\| = \sup_{\substack{\bar{U}x \neq 0,\\ x \in \mathscr{X}}} \frac{\|\mathrm{Proj}_N x\|}{\|\bar{U}x\|} = \sup \frac{\|\zeta\|}{\|\bar{U}\xi + \bar{U}\zeta\|} = \sup \frac{\|\bar{U}\zeta\|}{\|\bar{U}\xi + \bar{U}\zeta\|},$$

where $x = \xi + \zeta$, $\xi \in M$, $\zeta \in N$. As ξ may be zero, it follows that $\|q\| \geq 1$. On the other hand,

$$\|\bar{U}\xi + \bar{U}\zeta\|^2 = \|\bar{U}\xi\|^2 + \|\bar{U}\zeta\|^2,$$

since $(\xi, \zeta) = (\bar{U}\xi, \bar{U}\zeta) = 0$. Hence, $\|q\| = 1$.

Lemma 2 (dependence of spline on observation). *There is a unique map $e: F\mathscr{X} \to \mathscr{X}$ such that*

$$\prod = \mathrm{Proj}_M = eF;$$

e is linear, with closed graph. And e is continuous iff $F\mathscr{X}$ is closed in $\bar{Y}$. In any case

$$\|e\| \geq 1.$$

We may call e the *spline operator*. It carries the observation of $x \in \mathscr{X}$ into the approximation of x. Continuity of e is desirable because in practice Fx

is often known only approximately. If e is continuous, then e may be extended so as to be bounded linear on all of $\bar{Y}$ to $\mathscr{X}$, with no increase of norm. Then a contaminated observation $(\omega + \delta\omega) \in \bar{Y}$, where $\omega = \bar{F}x$, $x \in \mathscr{X}$, can be used as an input, and $e(\omega + \delta\omega) = e\omega + e(\delta\omega)$. The distortion $e(\delta\omega)$ is bounded in terms of $\|\delta\omega\|$. If $\bar{F}\mathscr{X}$ is finite dimensional, then e is surely continuous.

Proof of Lemma 2. Apply the quotient theorem with $\mathscr{A} = \mathscr{X} = \mathscr{C}$ and $\mathscr{B} = \bar{F}\mathscr{X}$. If $\bar{F}x = 0$, $x \in \mathscr{X}$, then $x \in N$ and $\mathrm{Proj}_M x = 0$. Hence, e is linear with closed graph; and e is continuous if $\mathscr{B}$ is closed.

We now show that conversely $\bar{F}\mathscr{X}$ is closed if e is continuous. As $\mathscr{X} = M \oplus N$ and $\bar{F}N = 0$, it follows that $\bar{F}\mathscr{X} = \bar{F}M$. Since e is continuous, there exists $b < \infty$ such that

$$\|ey\| \leq b\|y\|, \qquad y = \bar{F}x, \qquad x \in \mathscr{X}.$$

Now $e\bar{F}x = x$ if $x \in M$; therefore,

$$\|x\| \leq b\,\|\bar{F}x\|, \qquad x \in M.$$

To show that $\bar{F}M$ is closed, suppose that $\xi^\nu \in M$, $\nu = 1, 2,\ldots$, and that $\bar{F}\xi^\nu \to y \in \bar{Y}$ as $\nu \to \infty$. Then $\{\bar{F}\xi^\nu\}$ is a Cauchy sequence in $\bar{Y}$ and, therefore, $\{\xi^\nu\}$ is a Cauchy sequence in M. Hence, $\xi^\nu \to \xi \in M$. Since $\bar{F}$ is continuous, $\bar{F}\xi^\nu \to \bar{F}\xi = y \in \bar{F}M$. Thus, $\bar{F}M$ is closed.

It remains to show that $1 \leq \|e\| < \infty$. Now

$$\|e\|^2 = \sup_{\substack{\bar{F}x \neq 0,\\ x \in \mathscr{X}}} \frac{\|\xi\|^2}{\|\bar{F}x\|^2} = \sup_{\substack{\bar{F}\xi \neq 0,\\ \xi \in M}} \frac{\|\bar{F}\xi\|^2 + \|\bar{U}\xi\|^2}{\|\bar{F}\xi\|^2} \geq 1,$$

where $x = \xi + \zeta$, $\xi \in M$, $\zeta \in N$, since $\bar{F}x = \bar{F}\xi$. This completes the proof of the lemma. Note that $\|e\|$ will be finite iff $\|\bar{U}\xi\|/\|\bar{F}\xi\|$ is bounded, $\xi \in M$, $\bar{F}\xi \neq 0$.

Remark. Observation and coobservation are dual in our hypotheses but not in our construction or in the roles that they play. Thus, $N = \mathrm{kernel}\,\bar{F}$, whereas $M \supset \mathrm{kernel}\,\bar{U}$ properly. We envisage calculations based on a known $\bar{F}x$, with $\bar{U}x$ unknown or partially unknown, $x \in \mathscr{X}$.

4. Properties of Splines

THEOREM 1 (spline interpolation [5, 3, 14–16, 1]). *For each $x \in \mathscr{X}$ there is a unique $\xi \in M$ such that $\bar{F}\xi = \bar{F}x$; and $\xi = \prod x$.*

Proof. The condition $\bar{F}\xi = \bar{F}x$ may be written $x - \xi \in N$. The decomposition theorem for $\mathscr{X}$ implies that the decomposition of x into $\xi \in M$ and $x - \xi \in N$ is always possible and unique.

COROLLARY. *For each possible observation* $\omega \in \bar{F}\mathscr{X}$, *there is a unique* $\xi \in M$ *such that* $\bar{F}\xi = \omega$.

Proof. If $\omega = \bar{F}x^0$, then $\xi = \prod x^0 = e\omega$, by Lemma 2. Thus, ξ is unique.

LEMMA 3. *If* $x \in \mathscr{X}$ *and* $\xi = \prod x$, *then*

$$\|\bar{U}x\|^2 - \|\bar{U}\xi\|^2 = \|\bar{U}(x - \xi)\|^2 = \|x\|^2 - \|\xi\|^2 = \|x - \xi\|^2 = \|\operatorname{Proj}_N x\|^2$$

and

$$(\bar{U}\xi, \bar{U}x - \bar{U}\xi) = 0.$$

Proof. The last two of the continued equalities are immediate (Pythagoras). And

$$\|x - \xi\|^2 = \|\bar{F}(x - \xi)\|^2 + \|\bar{U}(x - \xi)\|^2 = \|\bar{U}(x - \xi)\|^2,$$

since $x - \xi \in N$. And

$$\|x\|^2 - \|\xi\|^2 = \|\bar{F}x\|^2 + \|\bar{U}x\|^2 - \|\bar{F}\xi\|^2 - \|\bar{U}\xi\|^2 = \|\bar{U}x\|^2 - \|\bar{U}\xi\|^2.$$

Finally,

$$(\xi, x - \xi) = 0 = (\bar{U}\xi, \bar{U}x - \bar{U}\xi).$$

THEOREM 2 (optimal interpolation [6, 5, 21, 3, 14–16, 1]). *For each* $x \in \mathscr{X}$, *the norm* $\|\bar{U}y\|$ *is minimal among all* $y \in \mathscr{X}$ *such that* $\bar{F}y = \bar{F}x$ *iff* $y = \prod x$.

Proof. Put $\xi = \prod x$. Then $\bar{F}\xi = \bar{F}x$. Now consider $y \in \mathscr{X}$ such that $\bar{F}y = \bar{F}x$, that is, $y - \xi \in N$. Then $\xi = \prod y$ and, by Lemma 3,

$$0 \leq \|y - \xi\|^2 = \|\bar{U}y\|^2 - \|\bar{U}\xi\|^2,$$

with equality iff $y = \xi$.

THEOREM 3 (approximation of $\bar{U}x$ [21, 3, 14, 16, 1]). *For each* $x \in \mathscr{X}$, *the norm* $\|\bar{U}(\eta - x)\|$ *is minimal among all* $\eta \in M$ *iff* $\bar{U}(\eta - \xi) = 0$, *where* $\xi = \prod x$.

Proof. If $\eta \in M$, then

$$\eta - x = (\eta - \xi) + (\xi - x), \qquad \eta - \xi \in M, \qquad \xi - x \in N;$$

$$\|\eta - x\|^2 = \|\eta - \xi\|^2 + \|\xi - x\|^2;$$

$$\|\bar{F}(\eta - x)\|^2 + \|\bar{U}(\eta - x)\|^2 = \|\bar{F}(\eta - \xi)\|^2 + \|\bar{U}(\eta - \xi)\|^2 + 0 + \|\bar{U}(\xi - x)\|^2.$$

Now $\bar{F}(\eta - x) = \bar{F}(\eta - \xi)$, since $x - \xi \in N$. Hence,

$$\|\bar{U}(\eta - x)\|^2 = \|\bar{U}(\eta - \xi)\|^2 + \|\bar{U}(\xi - x)\|^2 \geq \|\bar{U}(\xi - x)\|^2,$$

with equality iff $\bar{U}(\eta - \xi) = 0$.

THEOREM 4 (a lower bound on $\|\bar{U}x\|$ [17a; 12, p. 84]). *For any $x \in \bar{\mathscr{X}}$,*

$$\|\bar{U}x\| \geq \|\bar{U}e\bar{F}x\|$$

with equality iff $x = e\bar{F}x$.

Proof. Lemma 3 implies that $\|\bar{U}x\| \geq \|\bar{U}\xi\|$, $\xi = \prod x$, with equality iff $x = \xi$. Now $\xi = e\bar{F}x$, by Lemma 2. This completes the proof.

Note that the operator $\bar{U}e$ is accessible to us. Thus, Theorem 4 gives a lower bound on the norm of the coobservation in terms of the observation. If Y is finite dimensional, $\|\bar{U}e\bar{F}x\|^2$ is a quadratic form in the observation $\bar{F}x$.

It may appear surprising that $\bar{F}x$ should give information about $\bar{U}x$, as our sole hypothesis has been that $Fx = 0$, $Ux = 0$, $x \in X$ imply that $x = 0$. If F and U are independent, Theorem 4 will assert merely that $\|\bar{U}x\| \geq 0$ with equality iff $x = e\bar{F}x$. The more dependent F and U are, the more informative Theorem 4 is.

5. APPROXIMATION OF Gx

Suppose that G is a given operator on X to a normed linear space W. We now seek an approximation of G in terms of F. As the sets X and $\mathscr{X}$ are the same, G is an operator on $\mathscr{X}$ to W.

Assume that G has an extension $\bar{G}: \bar{\mathscr{X}} \to \bar{W}$ which is linear on $\bar{\mathscr{X}}$ and continuous on $N \subset \bar{\mathscr{X}}$. Let

$$J = \|\bar{G} \restriction N\|^2$$

be the square of the Banach norm of the restriction $\bar{G} \restriction N$ of $\bar{G}$ to N. Thus,

$$J = \sup_{\substack{\zeta \in N, \\ \|\zeta\| = 1}} \|\bar{G}\zeta\|^2 = \sup_{\substack{\zeta \in N, \\ \|\bar{U}\zeta\| = 1}} \|\bar{G}\zeta\|^2.$$

The last equality follows from the fact that $\|\zeta\|^2 = \|\bar{F}\zeta\|^2 + \|\bar{U}\zeta\|^2 = \|\bar{U}\zeta\|^2$.

Put

$$A_0 = \bar{G} \prod = \bar{G}e\bar{F}.$$

We shall see that A_0 is a natural approximation of G, among all maps which are independent of $\bar{U}$.

LEMMA 4. *For any* $x \in \mathscr{X}$,

$$\|\bar{G}x - A_0x\|^2 \leq J(\|\bar{U}x\|^2 - \|\bar{U}\xi\|^2) \leq J\|\bar{U}x\|^2, \tag{2}$$

where

$$\xi = \prod x = e\bar{F}x, \qquad A_0x = \bar{G}\xi.$$

The inequalities are sharp in the following strong sense. For each $\epsilon > 0$, *each* $\omega \in \bar{F}\mathscr{X}$ *and each* $d \geq \|\bar{U}e\omega\|$, *there is an* $x^0 \in \mathscr{X}$ *such that* $\bar{F}x^0 = \omega$, $\|\bar{U}x^0\| = d$, *and*

$$\|\bar{G}x^0 - A_0x^0\|^2 \geq J(\|\bar{U}x^0\|^2 - \|\bar{U}\xi^0\|^2) - \epsilon, \qquad \xi^0 = e\omega; \tag{3}$$

and for each $\epsilon > 0$ *and* $d \geq 0$, *there is an* $x^1 \in \mathscr{X}$ *such that* $\|\bar{U}x^1\| = d$ *and*

$$\|\bar{G}x^1 - A_0x^1\|^2 \geq J\|\bar{U}x^1\|^2 - \epsilon. \tag{4}$$

If G is a functional, then equality occurs in the first part of (2) *for any prescribed* $\omega = \bar{F}x \in \bar{F}\mathscr{X}$ *and* $d = \|\bar{U}x\| \geq \|\bar{U}e\omega\|$, *and equality occurs in both parts of* (2) *for any prescribed* $d = \|\bar{U}x\| \geq 0$.

Proof. Since $\bar{G} - A_0 = \bar{G}\,\mathrm{Proj}_N$, it follows that

$$\|\bar{G}x - A_0x\|^2 \leq J\|\mathrm{Proj}_N x\|^2 = J(\|\bar{U}x\|^2 - \|\bar{U}\xi\|^2), \qquad \xi = \prod x, \quad x \in \mathscr{X},$$

by Lemma 3 and the definition of J. This implies (2).

Suppose that $\epsilon > 0$, $\omega = \bar{F}x$, $x \in \mathscr{X}$, and $d \geq \|\bar{U}e\omega\|$ are prescribed. We shall show that $x^0 \in \mathscr{X}$ exists such that $\bar{F}x^0 = \omega$, $\|\bar{U}x^0\| = d$, and (3) holds.

If $d = \|\bar{U}e\omega\|$, this is immediate, for we take $x^0 = \xi$; then $\|\bar{G}x^0 - A_0x^0\| = 0 \geq -\epsilon$. Assume now that $d > \|\bar{U}e\omega\|$. Take k so that

$$|k|^2 = d^2 - \|\bar{U}e\omega\|^2.$$

The definition of J implies that $\zeta^0 \in N$ exists such that

$$\|\zeta^0\| = \|\bar{U}\zeta^0\| = 1$$

and

$$\|\bar{G}\zeta^0 - A_0\zeta^0\|^2 \geq J - \epsilon/|k|^2,$$

since $A_0\zeta^0 = \bar{G}\prod\zeta^0 = 0$. Put

$$x^0 = k\zeta^0 + \xi^0, \qquad \xi^0 = e\omega = \prod x.$$

Then $\prod x^0 = \xi^0$, and

$$\|\bar{U}x^0\|^2 - \|\bar{U}\xi^0\|^2 = \|\mathrm{Proj}_N x^0\|^2 = |k|^2\|\zeta^0\|^2 = |k|^2 = d^2 - \|\bar{U}\xi^0\|^2,$$

by Lemma 3. Hence, $\|\bar{U}x^0\| = d$, and $\bar{F}x^0 = \bar{F}\xi^0 = \bar{F}x = \omega$.

Now,

$$\bar{G}x^0 = k\bar{G}\zeta^0 + \bar{G}\xi^0,$$

$$A_0x^0 = kA_0\zeta^0 + A_0\xi^0 = kA_0\zeta^0 + \bar{G}\xi^0,$$

$$\|\bar{G}x^0 - A_0x^0\|^2 = |k|^2 \|\bar{G}\zeta^0 - A_0\zeta^0\|^2 \geq |k|^2(J - \epsilon/|k|^2) = |k|^2 J - \epsilon.$$

This establishes (3).

Next take $\omega = 0$ in the preceding discussion. Then (3) reduces to (4) with $x^1 = x^0$.

Finally, suppose that G is a functional. Then $\bar{G} \mid N$ is a linear continuous functional on the Hilbert space N. Let $g \in N$ be the dual (representer) of $\bar{G} \mid N$. Then, putting $\zeta^0 = g/\|g\|$, we see that

$$\|\zeta^0\| = \|\bar{U}\zeta^0\| = 1, \qquad \|\bar{G}\zeta^0 - A_0\zeta^0\|^2 = J.$$

The rest of the argument establishes (3) and (4) with $\epsilon = 0$ and inequality replaced by equality.

THEOREM 5 (geometric property [5, 15, 17, 2]). *For $\omega \in \bar{F}\mathscr{X}$ and $d \geq 0$, put*

$$\Gamma = \{x \in X: \bar{F}x = \omega \text{ and } \|\bar{U}x\| \leq d\}.$$

Then Γ is nonempty iff $d \geq \|\bar{U}e\omega\|$, and Γ is the intersection of the closed ball in $\mathscr{X}$ of radius $(\|\omega\|^2 + d^2)^{1/2}$, center 0, and the hyperplane $\xi^0 + N$, where

$$\xi^0 = e\omega \in M.$$

And $\bar{G}\Gamma$, if nonempty, is a convex bounded subset of $\overline{W}$, with center $\bar{G}\xi^0$ and maximum radius $J^{1/2}(d^2 - \|\bar{U}\xi^0\|^2)^{1/2}$.

As the center of a bounded set is unique, $\bar{G}\xi^0$ is a natural approximation of $\bar{G}x$, $x \in \Gamma$. But ξ^0 and $\bar{G}\xi^0$ are independent of d. Hence, $\bar{G}\xi^0$ is a natural approximation of $\bar{G}x$, for all $x \in \mathscr{X}$ for which $\bar{F}x = \omega$. As $\omega \in \bar{F}\mathscr{X}$ is arbitrary, $\bar{G}\xi$ is a natural approximation of $\bar{G}x$, for all $x \in \mathscr{X}$, where $\xi = \prod x = e\bar{F}x$.

Proof of Theorem 5. The set $\{x \in \mathscr{X}: \bar{F}x = \omega\}$ is $\xi^0 + N$, where $\omega = \bar{F}x^0$ for some $x^0 \in \mathscr{X}$ and $\xi^0 = \prod x^0 = e\omega$. Thus, $\bar{F}\xi^0 = \omega = \bar{F}(\xi^0 + N)$; and $\bar{F}x = \omega$, $x \in \mathscr{X}$, implies that $x - \xi^0 \in N$, and vice versa.

The set $\{x \in \mathscr{X}: \|x\|^2 \leq \|\omega\|^2 + d^2\}$ is the closed ball of square radius $\|\omega\|^2 + d^2$ and center 0. It is the set $\{x \in X: \|\bar{U}x\| \leq d\}$, since $\|x\|^2 = \|\omega\|^2 + \|\bar{U}x\|^2$.

Thus, Γ is the intersection of the two sets and is nonempty iff $\|\xi^0\|^2 \leq \|\omega\|^2 + d^2$, that is, $\|\bar{U}\xi^0\| \leq d$.

For any $x \in \Gamma$, $\prod x = e\omega = \xi^0$.

Since Γ is convex, so is $\bar{G}\Gamma$. We now show that $\bar{G}\Gamma$, if nonempty, has center $\bar{G}\xi^0$. Thus, consider any $x \in \Gamma$. Put $y = 2\xi^0 - x$. Then $\prod y = \xi^0$, and, by Lemma 3,

$$\| \bar{U}y \|^2 - \| \bar{U}\xi^0 \|^2 = \| \bar{U}(y - \xi^0)\|^2 = \| \bar{U}(x - \xi^0)\|^2 = \| \bar{U}x \|^2 - \| \bar{U}\xi^0 \|^2.$$

Hence, $\| \bar{U}y \| = \| \bar{U}x \|$. Also $\bar{F}y = 2\bar{F}\xi^0 - \bar{F}x = \bar{F}x = \omega$. Hence, $y \in \Gamma$, and $\bar{G}x, \bar{G}y \in \bar{G}\Gamma$. Now $\bar{G}y = 2\bar{G}\xi^0 - \bar{G}x$ and

$$\bar{G}y - \bar{G}\xi^0 = -(\bar{G}x - \bar{G}\xi^0).$$

Hence, $\bar{G}\Gamma$ has center $\bar{G}\xi^0$.

Lemma 4 implies that $\bar{G}\Gamma$ is bounded, with maximum square radius $\leqslant J(d^2 - \| \bar{U}\xi^0 \|^2)$ and $> J(d^2 - \| \bar{U}\xi^0 \|^2) - \epsilon$, for any $\epsilon > 0$. Hence, the maximum radius is as stated.

COROLLARY (optimality of A_0). *For each $\beta \in \bar{W}$, each $\omega \in \bar{F}\mathscr{X}$, and each $d \geqslant \| \bar{U}e\omega \|$,*

$$\sup_{x \in \Gamma} \| \bar{G}x - \beta \|^2 \geqslant J(d^2 - \| \bar{U}\xi^0 \|^2), \qquad \xi^0 = e\omega.$$

If $\bar{W}$ is a Hilbert space and $\beta \neq \bar{G}\xi^0$, then the supremum is strictly greater than the right member.

Thus, β, as an approximation of $\bar{G}x$, $x \in \Gamma$, is never better than $\bar{G}\xi^0 = A_0x$ (cf. Lemma 4). And if $\bar{W}$ is a Hilbert space, β is certainly worse, unless $\beta = \bar{G}\xi^0$.

Proof. By Lemma 4 there is a sequence $x^\nu \in \Gamma$, $\nu = 1, 2, \ldots$, such that

$$\| \bar{G}x^\nu - \bar{G}\xi^0 \|^2 > J(d^2 - \| \bar{U}\xi^0 \|^2) - 1/\nu,$$
$$\bar{F}x^\nu = \omega, \qquad \text{and} \qquad \| \bar{U}x^\nu \| = d.$$

Put

$$y^\nu = 2\xi^0 - x^\nu.$$

Then $y^\nu \in \Gamma$ for all ν. Let z^ν denote x^ν or y^ν according as $\| \bar{G}x^\nu - \beta \| \geqslant \| \bar{G}y^\nu - \beta \|$ or the contrary. Then

$$\begin{aligned} \| \bar{G}z^\nu - \beta \| &= \max(\| \bar{G}x^\nu - \beta \|, \| \bar{G}y^\nu - \beta \|) \\ &= \max(\|(\bar{G}x^\nu - \bar{G}\xi^0) + (\bar{G}\xi^0 - \beta)\|, \|(\bar{G}y^\nu - \bar{G}\xi^0) + (\bar{G}\xi^0 - \beta)\|) \\ &= \max(\|(\bar{G}\xi^0 - \beta) + (\bar{G}x^\nu - \bar{G}\xi^0)\|, \|(\bar{G}\xi^0 - \beta) - (\bar{G}x^\nu - \bar{G}\xi^0)\|). \end{aligned}$$

Now

$$\max(\| u + v \|, \| u - v \|) \geqslant \| v \|, \qquad u, v \in \bar{W}.$$

Hence,

$$\sup_{x\in\Gamma} \| \bar{G}x - \beta \|^2 \geqslant \sup_{\nu=1,2,\dots} \| \bar{G}x^\nu - \bar{G}\xi^0 \|^2 = J(d^2 - \| \bar{U}\xi^0 \|^2).$$

Furthermore, if $\bar{W}$ is a Hilbert space, $u \neq 0$, and $\| v \|$ is bounded, then

$$\max(\| u + v \|, \| u - v \|) - \| v \| \geqslant \delta$$

for some $\delta > 0$ [11, p. 230]. In this case, then,

$$\sup_{x\in\Gamma} \| \bar{G}x - \beta \|^2 \geqslant \sup_{\nu=1,2,\dots} \| \bar{G}x^\nu - \bar{G}\xi^0 \|^2 + \delta^2 > J(d^2 - \| \bar{U}\xi^0 \|^2),$$

if $\bar{G}\xi^0 - \beta \neq 0$, since $\| \bar{G}x^\nu - \bar{G}\xi^0 \|$ is bounded.

6. Admissible Approximations

Let $\mathscr{A}$ denote the set of operators $A: \mathscr{X} \to \bar{W}$ such that

$$A = E\bar{F} \qquad \text{and} \qquad R \underset{\text{def}}{=} \bar{G} - A = Q\bar{U},$$

where $E: \bar{F}\mathscr{X} \to \bar{W}$ and $Q: \bar{U}\mathscr{X} \to \bar{W}$ are linear. The algebraic part of the quotient theorem implies that $A \in \mathscr{A}$ iff Ax depends linearly on the observation $\bar{F}x$, $x \in \mathscr{X}$, and $Ax = \bar{G}x$ whenever $\bar{U}x = 0$. We say that A is an *admissible approximation of* $\bar{G}$ if $A \in \mathscr{A}$.

Now $A_0 = \bar{G} \prod$ is an admissible approximation of $\bar{G}$. For

$$A_0 = E_0\bar{F}, \qquad E_0 = \bar{G}e,$$

by Lemma 2; and

$$R_0 \underset{\text{def}}{=} \bar{G} - A_0 = \bar{G}\,\text{Proj}_N = Q_0\bar{U}, \qquad Q_0 = \bar{G}q,$$

by Lemma 1.

Theorem 6 (minimal[2] quotient [8; 9, Chapter 2; 11; 15; 17; 18–20; 4; 2]). *For $A \in \mathscr{A}$, the Banach norm $\| Q \|$ of Q is minimal if $A = A_0$, in which case*

$$\| Q \|^2 = J.$$

Conversely, if W is one-dimensional and $\| Q \|$ is minimal, then $A = A_0$. If W is many dimensional, then $\| Q \|$ may be minimal even though, $A \neq A_0$.

[2] Theorem 6 answers affirmatively a question raised in [9, p. 107] about broad and narrow interpolation. The spline approximation is best in both the broad and the narrow sense.

Proof. Part 1. Consider $A \in \mathscr{A}$ and $R = \bar{G} - A = Q\bar{U}$. From the definition of the Banach norm

$$\|Q\|^2 = \sup_{\substack{x\in\mathscr{X},\\ \bar{U}x\neq 0}} \frac{\|Rx\|^2}{\|\bar{U}x\|^2} \geq \sup_{\substack{\zeta\in N,\\ \bar{U}\zeta\neq 0}} \frac{\|R\zeta\|^2}{\|\bar{U}\zeta\|^2} = \sup_{\substack{\zeta\in N,\\ \|\bar{U}\zeta\|=1}} \|\bar{G}\zeta\|^2 = J,$$

since $A\zeta = 0$, $\zeta \in N$, and $\|\bar{U}\zeta\| = \|\zeta\|$. As $A_0 \in \mathscr{A}$, it follows that

$$\|Q_0\|^2 \geq J.$$

Now

$$\|Q_0\| = \sup_{\substack{x\in\mathscr{X},\\ \bar{U}x\neq 0}} \frac{\|R_0x\|}{\|\bar{U}x\|} = \sup_{\substack{x\in\mathscr{X},\\ \bar{U}x\neq 0,\\ \mathrm{Proj}_N x\neq 0}} \left(\frac{\|\bar{G}\,\mathrm{Proj}_N x\|}{\|\mathrm{Proj}_N x\|}\,\frac{\|\mathrm{Proj}_N x\|}{\|\bar{U}x\|}\right),$$

since the excluded case $\mathrm{Proj}_N x = 0$ would give $R_0x = 0$. By Lemma 3,

$$0 \leq \|\mathrm{Proj}_N x\|^2 = \|\bar{U}x\|^2 - \|\bar{U}\xi\|^2, \qquad \xi = \prod x, \quad x \in \mathscr{X},$$

and

$$0 \leq \frac{\|\mathrm{Proj}_N x\|^2}{\|\bar{U}x\|^2} = 1 - \frac{\|\bar{U}\xi\|^2}{\|\bar{U}x\|^2} \leq 1, \qquad \bar{U}x \neq 0.$$

Hence,

$$\|Q_0\|^2 \leq \sup_{\substack{x\in\mathscr{X},\\ \mathrm{Proj}_N x\neq 0}} \frac{\|\bar{G}\,\mathrm{Proj}_N x\|^2}{\|\mathrm{Proj}_N x\|^2} = \|\bar{G} \mid N\|^2 = J.$$

Thus,

$$\|Q_0\|^2 = J \leq \|Q\|^2$$

for all $A \in \mathscr{A}$.

Part 2. Suppose that W is one-dimensional. To fix our ideas, let the scalars be the complex numbers. We shall show that A_0 is the only element of $\mathscr{A}$ which minimizes $\|Q\|$; that is, that

$$\|Q\|^2 > J \qquad \text{if} \quad A_0 \neq A \in \mathscr{A}.$$

For all $\zeta \in N$, $R\zeta = \bar{G}\zeta = R_0\zeta$. Hence, there must be an element $\xi^0 \in M$ such that $R\xi^0 \neq 0$ (otherwise $R = R_0$). Then $\bar{U}\xi^0 \neq 0$, since $R = Q\bar{U}$.

Now

$$J = \sup_{\substack{\zeta\in N,\\ \|\bar{U}\zeta\|=1}} \|G\zeta\|^2.$$

If $J = 0$, then $\bar{G}\zeta = 0$ for all $\zeta \in N$, and $R_0x = \bar{G}\,\mathrm{Proj}_N x = 0$ for all $x \in \mathscr{X}$. Hence, $Q_0 = 0$. On the other hand, $Q \neq 0$, else $R\xi^0 = 0$. Hence, $\|Q\| > \|Q_0\| = 0$, as was to be shown. Now assume that $J > 0$.

Since $\bar{G} \upharpoonright N$ is a linear continuous functional on the Hilbert space N, there exists $\zeta^0 \in N$ such that

$$\| \bar{U}\zeta^0 \| = \| \zeta^0 \| = 1, \qquad \bar{G}\zeta^0 = J^{1/2} > 0.$$

Put

$$x^0 = \frac{\overline{R\xi^0}}{J^{1/2} \| \bar{U}\xi^0 \|^2} \xi^0 + \zeta^0 \in \bar{\mathscr{X}}.$$

Then

$$Rx^0 = \frac{| R\xi^0 |^2}{J^{1/2} \| \bar{U}\xi^0 \|^2} + J^{1/2} = \frac{| R\xi^0 |^2 + J \| \bar{U}\xi^0 \|^2}{J^{1/2} \| \bar{U}\xi^0 \|^2} = | Rx^0 |.$$

And

$$\| \bar{U}x^0 \|^2 = \frac{| R\xi^0 |^2}{J \| \bar{U}\xi^0 \|^4} \| \bar{U}\xi^0 \|^2 + \| \bar{U}\zeta^0 \|^2 = \frac{| R\xi^0 |^2 + J \| \bar{U}\xi^0 \|^2}{J \| \bar{U}\xi^0 \|^2},$$

since $(\bar{U}\xi^0, \bar{U}\zeta^0) = (\xi^0, \zeta^0) = 0$ and $\| \bar{U}\zeta^0 \| = 1$. Hence,

$$\frac{| Rx^0 |^2}{\| \bar{U}x^0 \|^2} = \frac{| R\xi^0 |^2 + J \| \bar{U}\xi^0 \|^2}{\| \bar{U}\xi^0 \|^2} = J + \frac{| R\xi^0 |^2}{\| \bar{U}\xi^0 \|^2} > J.$$

Hence,

$$\| Q \|^2 > J.$$

Part 3. If W is many dimensional, the following elementary example shows that A_0 need not be the only element of $\mathscr{A}$ which minimizes $\| Q \|$.

Let

$$\begin{aligned} X &= \mathbb{R}^5, \qquad Y = Z = W = \mathbb{R}^3; \\ Fx &= (x_3, x_4, x_5) \in Y, \\ Ux &= (x_1, x_2, x_3) \in Z, \\ Gx &= (x_1, x_2, x_4) \in W, \\ x &= (x_1, x_2, x_3, x_4, x_5) \in X. \end{aligned}$$

Completions of spaces and operators are not needed because of the finite dimensionality. Now [11, p. 238], $A \in \mathscr{A}$ iff

$$Ax = (ax_3, bx_3, cx_3 + x_4), \qquad a, b, c \in \mathbb{R}.$$

Then

$$Rx = (x_1 - ax_3, x_2 - bx_3, -cx_3) = QUx,$$

where $Q: Z \to W$ is represented by the 3×3 matrix

$$\begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & -c \end{pmatrix}.$$

Now $\|Q\|^2$ is the largest absolute autovalue of the product of this matrix by its transpose. And 1 is an autovalue. Hence,

$$\|Q\| \geqslant 1.$$

Clearly $\|Q\| = 1$, if $a = b = 0$ and $|c| \leqslant 1$. Thus, the minimum is not attained uniquely. The reader may verify that $Q = Q_0$ iff $a = b = c = 0$.

7. The Completeness Condition

In some applications the space X is normed and the norms in X and $\mathscr{X}$ are equivalent. Then X and $\mathscr{X}$ have the same topology and $\overline{X} = \bar{\mathscr{X}}$. The simplest case is the one in which $X = \overline{X} = \bar{\mathscr{X}}$; all the bars over our symbols may then be omitted.

Let us say that the completeness condition [5] holds if X is normed, if F and U are continuous on X, and if $b < \infty$ exists such that

$$\|x\|_X^2 \leqslant b^2(\|Fx\|^2 + \|Ux\|^2), \qquad \text{all} \quad x \in X.$$

This implies that kernel $F \cap$ kernel $U = \{0\}$.

Lemma 5. *Suppose that the completeness condition holds. Then X and $\mathscr{X}$ have equivalent norms. As sets $X = \mathscr{X}$ and $\overline{X} = \bar{\mathscr{X}}$. Furthermore, if $G\colon X \to W$ is linear and continuous, then the completion $\bar{G}\colon \bar{\mathscr{X}} \to \overline{W}$ exists and is linear and continuous. Conversely, if X is normed, if F, U are continuous on X, and $\overline{X} = \bar{\mathscr{X}}$, then the completeness condition holds.*

Proof. Since F and U are continuous on X, $c < \infty$ exists such that

$$\|x\|_{\mathscr{X}}^2 = \|Fx\|^2 + \|Ux\|^2 \leqslant c^2\|x\|_X^2, \qquad \text{all} \quad x \in X = \mathscr{X}.$$

Now the completeness condition implies that

$$\|x\|_X^2 \leqslant b^2\|x\|_{\mathscr{X}}^2.$$

Thus, X and $\mathscr{X}$ have equivalent norms. Hence, the completions $\overline{X}$ and $\bar{\mathscr{X}}$ are equal as sets and have equivalent norms. Finally, $\bar{G}\colon \bar{\mathscr{X}} \to \overline{W}$ is linear and continuous since $G\colon X \to W$ is.

To prove the converse statement, note that if $\overline{X} = \bar{\mathscr{X}}$, then the identity: $\bar{\mathscr{X}} \to \overline{X}$ is continuous, by one of Banach's theorems [9, p. 307], and the completeness condition holds.

8. Harmonic Functions

In the following application of the theory, harmonic functions are splines.

Let Ω be an open region of $\mathbb{R}^m$ on which the divergence theorem holds, and let β be the boundary of Ω. Then β is an admissible domain of integration of an $(m-1)$-fold integral in $\mathbb{R}^m$.

Let X be the set of C_2 functions on the closure $\bar{\Omega}$. Thus, $x \in X$ iff $x: \bar{\Omega} \to \mathbb{R}$ has an extension which is C_2 on a neighborhood of $\bar{\Omega}$.

Let $Y = L^2(\beta)$. Thus, $y \in Y$ means that $y: \beta \to \mathbb{R}$ is Hausdorff $(m-1)$-measurable and that $\int_\beta |y|^2 < \infty$, with the usual convention that y need be defined only almost everywhere $(m-1)$ on β and that two functions which are equal almost everywhere $(m-1)$ on β correspond to the same element of Y. Also

$$(x, y) = \int_\beta xy, \qquad x, y \in Y.$$

The integral here is relative to $(m-1)$-measure.

Let $Z = L^2(\Omega) \times L^2(\Omega) \times \cdots$ to m factors, where $L^2(\Omega)$ is the usual L^2 space on Ω. If $x = (x^1, \ldots, x^m)$ and $y = (y^1, \ldots, y^m)$ are elements of Z, then

$$(x, y) = \int_\Omega \int \sum_{j=1}^{m} x^j y^j.$$

The integral here is relative to Lebesgue measure in $\mathbb{R}^m$. We shall use double and single integral signs to indicate m-fold and $(m-1)$-fold integrals, over the domains Ω and $\beta = \partial\Omega$, respectively (unless other domains are indicated explicitly).

Let $F: X \to Y$ be the operator of restriction to β, so that $Fx = x \upharpoonright \beta$. Since $x \upharpoonright \beta$ is continuous, it is surely an element of Y. The observation of x is in effect the set of boundary values of x.

Let $U: X \to Z$ be the gradient operator. Thus,

$$Ux = \operatorname{grad} x = [x_1, \ldots, x_m],$$

where subscripts indicate partial derivatives. The coobservation of x is its gradient. And

$$(Ux, Uy) = \iint \operatorname{grad} x \cdot \operatorname{grad} y = \iint (x_1 y_1 + \cdots + x_m y_m).$$

Now $Fx = 0$, $Ux = 0$, $x \in X$ imply that $x = 0$. For $Ux = 0$ implies that x is locally constant, hence constant on each connected component of $\bar{\Omega}$. And $Fx = 0$ implies that the constant is 0. Thus, we may and shall consider splines relative to F and U.

The space $\mathscr{X}$ is X with the inner product

$$(x, y) = \int xy + \iint \operatorname{grad} x \cdot \operatorname{grad} y, \qquad x, y \in \mathscr{X};$$

$\bar{\mathscr{X}}$ is the completion of $\mathscr{X}$. The completions of F, U are $\bar{F}$, $\bar{U}$. Thus, for example, $\bar{y} = \bar{F}\bar{x}$, $\bar{x} \in \bar{\mathscr{X}}$, means that there is a sequence $x^\nu \in \mathscr{X}$, $\nu = 1, 2, \ldots,$

such that $x^\nu \to \bar{x}$ as $\nu \to \infty$ and $Fx^\nu \to \bar{y} \in \bar{Y} = L^2(\beta)$. We describe the situation informally by saying that $\bar{x} \mid \beta = \bar{y}$. Similarly, if $\Xi = \bar{U}\bar{x}$, we say that grad $\bar{x} = \Xi$.

Now

$$N = \{x \in \bar{\mathscr{X}} : x \mid \beta = 0\}$$

consists of the elements of $\bar{\mathscr{X}}$ which vanish almost everywhere on the boundary of Ω, and

$$M = \left\{x \in \bar{\mathscr{X}} : \iint \text{grad } x \cdot \text{grad } \zeta = 0 \text{ whenever } \zeta \in N\right\}.$$

Green's first formula is

$$\iint (\text{grad } x \cdot \text{grad } y + y \text{ lap } x) = \int yn \cdot \text{grad } x, \qquad x, y \in \mathscr{X},$$

where lap $x = x_{1,1} + \cdots + x_{m,m}$ and n is the unit normal of β. This implies that *harmonic functions in* $\mathscr{X}$ *are splines and, conversely, elements of* $M \cap \mathscr{X}$ *are harmonic*. For, suppose that $x \in \mathscr{X}$ and lap $x = 0$. Consider any $\zeta \in N$. Then there is a sequence $\zeta^\nu \in \mathscr{X}$, $\nu = 1, 2, \ldots$, such that $\zeta^\nu \to \zeta$ and $\zeta^\nu \mid \beta \to 0$ as $\nu \to \infty$. Now

$$\iint \text{grad } x \cdot \text{grad } \zeta^\nu = \int \zeta^\nu n \cdot \text{grad } x = \int \zeta^\nu \mid \beta n \cdot \text{grad } x \to 0.$$

Hence,

$$\iint \text{grad } x \cdot \text{grad } \zeta = 0$$

and $x \in M$. Conversely, if $x \in M \cap \mathscr{X}$, then

$$\iint \zeta \text{ lap } x = 0$$

for all $\zeta \in N \cap \mathscr{X}$. Since $x \in \mathscr{X}$, lap x is continuous and, therefore, vanishes on Ω.

As the elements of $M \cap \mathscr{X}$ are harmonic functions, it is natural to call the elements of $M \cap \bar{\mathscr{X}} = M$ *generalized* harmonic functions. We shall do this. Thus, splines relative to the present F and U are generalized harmonic functions.

Theorem 1 now states that there is one and only one generalized harmonic function with prescribed boundary values. The generalized Dirichlet problem has one and only one generalized solution.

Theorem 2 states that $\iint \|\operatorname{grad} x\|^2$ has a minimum among all $x \in \bar{\mathscr{X}}$ with prescribed boundary values, that the minimum occurs uniquely, and that the minimizing x is a generalized harmonic function. This is the generalized Dirichlet principle.

Theorem 5 implies that for any $x \in \bar{\mathscr{X}}$, among functions that agree with x on the boundary, the generalized harmonic function is the best approximation.

The spline operator e of Lemma 2 is the known integral operator, whose kernel is the normal derivative of Green's function, which produces the harmonic function having specified boundary values. If Ω has a Green's function with suitable properties, then e is continuous.

9. Other Applications

(i) Let X, Z and $U: X \to Z$ be as in the preceding section. For $x \in X$, let Fx be something more than $x \restriction \beta$. For example, Fx may be the triple $(x \restriction \beta, x \restriction \mathscr{D}, \iint_{\mathscr{E}} x)$, where $\mathscr{D}$ and $\mathscr{E}$ are preassigned subsets of Ω. The essential point for our theory is that $Fx \in Y$ and Y be an inner product space. Here Y may be $L^2(\beta) \times L^2(\mathscr{D}) \times \mathbb{R}$, so that

$$(x, y) = \int_\beta xy + \int_{\mathscr{D}}\!\!\int xy - \left(\int_{\mathscr{E}}\!\!\int x\right)\left(\int_{\mathscr{E}}\!\!\int y\right), \qquad x, y \in Y.$$

The present Fx contains more information than that of the preceding section. Hence, $Fx = 0$, $Ux = 0$, $x \in X$ imply that $x = 0$. We may, therefore, apply our theory. The space N will be smaller than before, and, therefore, M will be larger. The splines in the present application constitute a stronger tool than do the generalized harmonic functions, but a tool which requires more complicated calculations.

(ii) We may use higher derivatives. With X as before, a possible coobservation is the second derivative

$$Ux = D^2x: \Omega \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R},$$

where Z is the space with inner product

$$(Ux, Uy) = \iint \sum_{i,j} x_{i,j} y_{i,j}, \qquad x, y \in X,$$

and $x_{i,j}$ is the partial derivative $\partial^2 x/\partial s^i \partial s^j$, $(s^1,\ldots, s^m) \in \Omega$. The observation must be such that kernel $F \cap$ kernel $U = \{0\}$.

(iii) Even in the analysis of functions of one variable, there may be interesting applications involving nonscalar observations. One elementary instance, perhaps suggestive, is the following.

Let a be the characteristic function of the interval [0, 2]:

$$a(s) = \begin{cases} 1 & \text{if } 0 \leqslant s \leqslant 2, \\ 0 & \text{otherwise;} \end{cases}$$

and b the characteristic function of [1, 3]:

$$b(s) = \begin{cases} 1 & \text{if } 1 \leqslant s \leqslant 3, \\ 0 & \text{otherwise.} \end{cases}$$

Let $X = C_0[0, 3]$ = space of continuous functions on [0, 3] to $\mathbb{R}$. Let $Y = L^2[0, 2]$, and $F: X \to Y$ be the operator $Fx = ax$ = restriction of x to [0, 2]. Let $Z = L^2[1, 3]$, and $U: X \to Z$ be the operator $Ux = bx$ = restriction of x to [1, 3]. Then $Fx = 0$, $Ux = 0$, $x \in X$ imply that x vanishes almost everywhere on [0, 3], hence, that $x = 0$. We may, therefore, apply our theory.

The inner product in $\mathscr{X}$ is

$$(x, y) = \int_0^2 xy + \int_1^3 xy = \int xy\, d\mu, \qquad x, y \in \mathscr{X},$$

where

$$d\mu = \begin{cases} 2dx & \text{on } [1, 2], \\ dx & \text{on } [0, 1) \text{ and } (2, 3], \\ 0 & \text{elsewhere.} \end{cases}$$

Hence, $\mathscr{X} = X \cap L^2(\mu)$. As X is dense in $L^2(\mu)$, it follows that

$$\bar{\mathscr{X}} = L^2(\mu).$$

Now $\bar{Y} = Y$, $\bar{Z} = Z$, and

$$\bar{F}x = ax, \qquad \bar{U}x = bx, \qquad x \in \bar{\mathscr{X}}.$$

Next

$$N = \text{kernel}\, \bar{F} = \{x \in \bar{\mathscr{X}}: ax = 0\} = \{x \in \bar{\mathscr{X}}: x = 0 \text{ a.e. on } [0, 2]\}.$$

$$M = N^{\perp} = \left\{\xi \in \bar{\mathscr{X}}: \int_1^3 b\xi b\zeta = 0 \text{ whenever } \zeta \in \bar{\mathscr{X}} \text{ vanishes a.e. on } [0, 2]\right\}$$

$$= \left\{\xi \in \bar{\mathscr{X}}: \int_2^3 \xi\zeta = 0 \text{ whenever } \zeta \in L^2[2, 3]\right\}$$

$$= \{\xi \in \bar{\mathscr{X}}: \xi = 0 \text{ a.e. on } [2, 3]\} = \{\xi \in \bar{\mathscr{X}}: (1 - a)\, \xi = 0\}.$$

Also,

$$\text{Proj}_M\, x = ax, \qquad \text{Proj}_N\, x = (1 - a)x, \qquad x \in \bar{\mathscr{X}},$$

since $x = ax + (1 - a)x$, and $ax \in M$, $(1 - a)x \in N$.

Thus, the optimal approximation of $x \in \bar{\mathscr{X}}$ is ax, as one would expect, and the optimal approximation of $\bar{G}x$ is $\bar{G}(ax)$, where $G: \mathscr{X} \to W$ is linear continuous and W is a normed linear space. One may calculate the operators q and e of Lemmas 1 and 2. Indeed,

$$qy = (1 - a)y, \quad y \in \overline{U\mathscr{X}}; \qquad \text{and} \qquad e = \text{identity}: \bar{F}\bar{\mathscr{X}} \to \bar{\mathscr{X}};$$

and

$$\|q\| = 1, \qquad \|e\|^2 = 2.$$

The reader may construct similar applications in which observation and coobservation involve derivatives or integrals.

In any application of our theory, the interchange of observation and coobservation produces a new application.

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